Lecture 33

10.2 - Calculus with Parametric Curves

Suppose C is a parametric curve described by the parametric equations

$$x = f(t), \quad y = g(t).$$

Using the chain rule, we can find $\frac{dy}{dx}$ & $\frac{dx}{dy}$ in terms of $\frac{dx}{dt}$ & $\frac{dy}{dt}$:

$\boxed{\frac{dy}{dx}}$ $\boxed{\frac{dx}{dy}}$

As usual, $\frac{dy}{dx}$ will give us the slope of the tangent line to the curve at a given point. The descriptions of $\frac{dy}{dx}$ & $\frac{dx}{dy}$ in terms of $\frac{dx}{dt}$ & $\frac{dy}{dt}$ can help us to find vertical and horizontal tangents:

Recall that a curve has a horizontal tangent when $\frac{dy}{dx} = 0$. Using the above, we see this is the same as

$$\frac{dy}{dt} = \qquad \left(\frac{dx}{dt} = \qquad \right)$$

A vertical tangent happens when $\frac{dy}{dx} = \infty$, or $\frac{dx}{dy} = 0$. Using parametric equations, this becomes:

$$\frac{dx}{dt} = \qquad \left(\frac{dy}{dt} = \qquad \right)$$

We can use the chain rule again to find second derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) =$$

which is valid as long as:

This lets us find where parametric curves are concave up or down at.

Ex: Consider the curve with parametric equations

$$x = t^3 - 3t, \quad y = t^2 - 3$$

ⓐ Find the points on the curve where the tangent is vertical.

ⓑ At what points is the tangent horizontal?

ⓒ Find an equation for the tangent line at $t = 2$.

(d) For what t-values is the graph concave up? concave down?

Parametric curves can pass through the same point more than once... an not necessarily in the same way.

Ex: Find the slope of both tangent lines to

$$x = t\cos t, \quad y = t\sin t, \quad -\pi \le t \le \pi$$

at the point $(0, \frac{\pi}{2})$.

Area Under a Curve

We know from calc I that the area under a curve $y=F(x)$, $a \leq x \leq b$, and $F(x)>0$ is given by

$$\int_a^b F(x)\,dx$$

but sometimes this integral is tedious or impossible to compute. Suppose we parametrize this curve by

$$x=f(t),\ y=g(t),\quad \alpha \leq t \leq \beta$$

Then, since $y=F(x)$

$$\int_a^b F(x)\,dx = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,d(f(t))$$

$$= \int_\alpha^\beta g(t) f'(t)\,dt$$

*It might be necessary to integrate from β to α... it depends on which direction the curve is parametrized in.

Area is given by

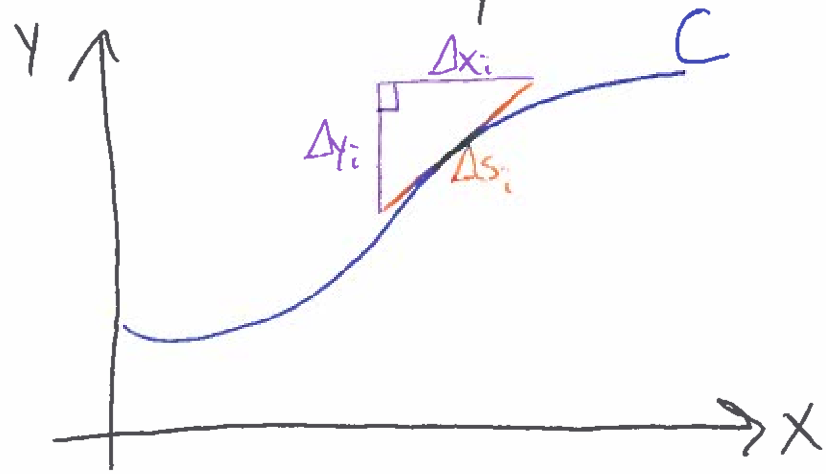
$$\int_\alpha^\beta g(t)f'(t)\,dt \quad \text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt$$

Ex: Find the area under the curve

$$x = 2\cos t,\quad y = 3\sin t,\quad 0 \le t \le \frac{\pi}{2}$$

Arc Length

Recall from chapter 8:



$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

So, the length of the curve C is given by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \int_C ds$$

If C has parametric equations

$$x=f(t),\ y=g(t),\quad a\le t\le b$$

then

$$L=\int_C ds=\int_a^b\sqrt{(dx)^2+(dy)^2}$$

$$=\int_a^b\sqrt{[d(f(t))]^2+[d(g(t))]^2}$$

$$=\int_a^b\sqrt{[f'(t)\,dt]^2+[g'(t)\,dt]^2}$$

$$=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$$

$$=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

There are some conditions need for this though:

1) $f'(t)$ & $g'(t)$ are continuous on $[a,b]$

2) The curve C is traversed <u>exactly once</u> as t increases from a to b.

Ex: Show that the circumference of a circle of radius r is indeed $2\pi r$.

Ex: Find the arclength of the circle defined by

$$x = \cos 2t,\ y = \sin 2t,\ 0 \le t \le 2\pi.$$

What is the significance of this?